Journal of Applied Mechanics and Technical Physics, Vol. 37, No. 3, 1996

TWO-PARAMETER FAMILY OF SUCCESSIVE (M, N)-APPROXIMATIONS OF THE EQUATIONS OF AN ELASTIC LAYER OF VARIABLE THICKNESS

A. E. Alekseev

UDC 539.3

One method for deriving the two-dimensional equations of the theory of plates and shells is the method of expansion in terms of thickness using Legendre's polynomials (for example, [1]). Ivanov [2] proposed a method based on the use of several approximations of the same unknown functions as truncated series in terms of Legendre's polynomials. Developing this technique, Alekseev [3] derived a one-parameter family of successive approximations of the equations of deformation of a layer of variable thickness in an arbitrary curvilinear coordinate system. For a layer of constant thickness in an orthogonal curvilinear coordinate system successive approximations that depend on two parameters and are called (M, N)-approximations are reported by Pelekh et al. [4]. Below, we propose a method, which is different from that proposed in [4], for reducing the three-dimensional equations of the theory of elasticity to a two-parameter sequence of two-dimensional problems of an elastic layer of variable thickness in an arbitrary curvilinear coordinate system. This method develops the results of [2, 3].

1. Definition of the Geometry of the Layer. We denote by V the region of three-dimensional space R^3 occupied by a shell. We define the position of the faces S^+ and S^- by specifying the radius vectors $\mathbf{R}^+$ and $\mathbf{R}^-$ as functions of the same Gaussian coordinates ξ^α:

$$\mathbf{R}^+ = \mathbf{R}^+(\xi^\alpha), \qquad \mathbf{R}^- = \mathbf{R}^-(\xi^\alpha), \qquad \{\xi^\alpha\} \in S_\xi \subset R^2.$$

Hereafter the Greek superscripts and subscripts take values 1 and 2, and the Latin superscripts and subscripts, 1, 2, and 3.

The functions $\mathbf{R}^+$ and $\mathbf{R}^-$ map a plane region S_ξ with a boundary L_ξ in space R^2 onto the faces S^+ and S^-, respectively. The position of each internal point of the shell V is defined by a vector function of the curvilinear coordinates ξ^k:

$$\mathbf{R}(\xi^k) = \mathbf{r}_0(\xi^\alpha) + \xi^3 \Delta\mathbf{r}(\xi^\alpha), \qquad \{\xi^k\} \in V_\xi \subset R^3, \tag{1.1}$$

where

$$V_\xi = \{\xi^k | \xi^\alpha \in S_\xi \subset R^2,\ \ \xi^3 \in [-1, 1]\}, \quad \mathbf{r}_0 = 0.5(\mathbf{R}^+(\xi^\alpha) + \mathbf{R}^-(\xi^\alpha)), \quad \Delta\mathbf{r} = 0.5(\mathbf{R}^+(\xi^\alpha) - \mathbf{R}^-(\xi^\alpha)). \tag{1.2}$$

In this case, the vector function $\mathbf{R}$ maps V_ξ onto V, and the vector function $\mathbf{r}_0$ maps the plane region S_ξ onto the surface S_0 in three-dimensional space, which is called below the middle surface.

Let h denote half the layer thickness along ξ^3. From (1.2), we obtain $h = (\Delta\mathbf{r} \cdot \Delta\mathbf{r})^{0.5}$, $\Delta\mathbf{r} = h\mathbf{n}$ ($\mathbf{n}$ is a unit vector along ξ^3).

Let Σ be the side surface of the shell, and L, the line of intersection of Σ and S_0. Then, following (1.2), Σ is a ruled surface formed by a family of straight lines passing through points of the boundary L in the direction of the vector $\mathbf{n}$. We fix a point with coordinates $\{\xi_0^\alpha\}$ in the region S_ξ. Then, following (1.1),

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 37, No. 3, pp. 133–144, May–June, 1996. Original article submitted March 2, 1995.

0021-8944/96/3703-0411 $15.00

the vector function $\mathbf{R}$ associates this point with a segment of a straight line in three-dimensional space that ends on the faces. Moreover, if this point belongs to L, the entire segment belongs to the side surface Σ.

Thus, the geometry of a shell of variable thickness is completely specified by the vector functions $\mathbf{R}^+$ and $\mathbf{R}^-$. In this case, the vector $\mathbf{n}$ is not necessarily normal to the surface S_0, and the side surface Σ and the middle surface S_0 may not intersect at a right angle.

2. Local Bases of the Coordinate System of the Layer. Following (1.1), one can take a set of three numbers $\{\xi^k\}$ as the coordinates of each point of the layer V; $\{\xi^\alpha\}$ are the Gaussian coordinates of the middle surface S_0, and $\xi^3 \in [-1, 1]$ is the coordinate along $\mathbf{n}$. Such a curvilinear coordinate system is called the coordinate system of the layer.

Differentiating both sides of Eq. (1.1) with respect to the variables ξ^k, we obtain the vector functions

$$\mathbf{э}_\alpha = \mathbf{R}_{,\alpha} = \left(\mathbf{R}_{,\alpha} = \frac{\partial \mathbf{R}}{\partial \xi^\alpha}\right) = \mathbf{r}_{0,\alpha} + \Delta \mathbf{r}_{,\alpha}\xi^3, \qquad \mathbf{э}_3 = \mathbf{R}_{,3} = \Delta \mathbf{r} = h\mathbf{n}, \tag{2.1}$$

which form a covariant local basis for the coordinate system of the layer.

Let us introduce the following notation: $\mathbf{э}^0_\alpha = \mathbf{r}_{0,\alpha} = \mathbf{э}_\alpha(\xi^\beta, 0)$, $\mathbf{э}^0_3 = \mathbf{э}_3(\xi^\alpha)$, and $\mathbf{э}^{0i} = \mathbf{э}^i(\xi^\alpha, 0)$.

The triples of the vectors $\mathbf{э}^0_i$ and $\mathbf{э}^{0i}$ form local bases on the middle surface of the layer. The local bases at an arbitrary point of the layer are determined by parallel translation from the corresponding point of the middle surface.

For each point of the layer we determine a triple of vectors $\mathbf{a}_i$,

$$\mathbf{a}_\alpha = \mathbf{n} \times (\mathbf{э}^0_\alpha \times \mathbf{n}), \qquad \mathbf{a}_3 = \mathbf{n}, \tag{2.2}$$

and consider it further as the main local (covariant) basis of the layer. Following the well-known formula from vector algebra, from (2.2) we find $\mathbf{a}_\alpha = \mathbf{э}^0_\alpha - \mathbf{n}\cdot(\mathbf{э}^0_\alpha \cdot \mathbf{n})$, i.e., the vector $\mathbf{a}_\alpha$ is the projection of $\mathbf{э}^0_\alpha$ onto the plane orthogonal to the unit vector $\mathbf{n}$. The corresponding biorthogonal (contravariant) vector is found from the conditions $\mathbf{a}_i \cdot \mathbf{a}^j = \delta^j_i$ (δ^j_i is the Kronecker symbol) and is of the form $\mathbf{a}^\alpha = \mathbf{э}^{0\alpha}$, $\mathbf{a}^3 = \mathbf{n}$. It follows from (2.2) that the vector $\mathbf{n}$ is orthogonal to the base vectors $\mathbf{a}^\alpha$ and $\mathbf{a}_\alpha$.

Thus, using the above method, we can define three types of local bases for the curvilinear coordinate system of the layer ($\mathbf{э}_i$, $\mathbf{э}^0_i$, and $\mathbf{a}_i$). Every vector from one basis can be represented as a linear combination of vectors of another basis. Denote

$$g_{ij} = \mathbf{э}_i \cdot \mathbf{э}_j, \qquad g^0_{ij} = \mathbf{э}^0_i \cdot \mathbf{э}^0_j, \qquad a_{ij} = \mathbf{a}_i \cdot \mathbf{a}_j, \tag{2.3}$$

where g_{ij} are the components of the metric tensor of the coordinate system ξ^i. It follows from (2.1) and (2.3) that $g_{33} = h^2$, and from (2.2) we have $a_{33} = 1$ and $a_{3\alpha} = 0$.

Using the notation $g_\alpha = g_{\alpha 3}/h$ and $b^\beta_\alpha = -(\mathbf{a}_{3,\alpha} \cdot \mathbf{a}^\beta)$, we write formulas (2.1) as

$$\mathbf{э}_\alpha = m^\beta_\alpha \mathbf{a}_\beta + g_\alpha \mathbf{a}_3, \qquad \mathbf{э}_3 = h\mathbf{a}_3 \qquad (m^\beta_\alpha = \delta^\beta_\alpha - b^\beta_\alpha h\xi^3).$$

Accordingly, for components of the metric tensor g_{ij} we have

$$g_{\alpha\beta} = m^\gamma_\alpha m^\lambda_\beta a_{\gamma\lambda} + g_\alpha g_\beta, \qquad g_{\alpha 3} = hg_\alpha, \qquad g_{33} = h^2. \tag{2.4}$$

From (2.4) we obtain formulas that relate the determinants g, g^0, and a of the matrices $\|g_{ij}\|$, $\|g^0_{ij}\|$, and $\|a_{ij}\|$: $g = h^2 m^2 a$ and $g^0 = h^2 a$, ($m = m^1_1 m^2_2 - m^2_1 m^1_2$ is the determinant of the matrix $\|m_{\alpha\beta}\|$).

3. Equations of the Linear Theory of Elasticity in an Arbitrary Curvilinear Coordinate System. Let us consider an arbitrary curvilinear coordinate system ξ^i. The equilibrium equations for a continuum are written in vector form [1] as

$$\hat{\mathbf{t}}^i_{,i} + \hat{\mathbf{f}} = 0, \qquad \hat{\mathbf{t}}^i = J\mathbf{t}^i, \qquad \hat{\mathbf{f}} = J\mathbf{f}, \qquad \mathbf{t}^i = \sigma^{ij}\mathbf{э}_j; \tag{3.1}$$

$$\mathbf{э}_i \times \hat{\mathbf{t}}^i = 0, \qquad J = \mathbf{э}_1 \cdot (\mathbf{э}_2 \times \mathbf{э}_3), \tag{3.2}$$

where J is the Jacobian of transformation of the coordinates; σ^{ij} are the components of the stress tensor; $\mathbf{f}$ is the vector of volume forces. Equation (3.2) is the condition of symmetry of the stress tensor.

The components of the strain tensor ε_{ij} are related to the displacement vector $\mathbf{u}$ by the linear relations

$$2\varepsilon_{ij} = (\ni_i \cdot \mathbf{u}_{,j}) + (\ni_j \cdot \mathbf{u}_{,i}). \tag{3.3}$$

The generalized Hooke's law has the form

$$\sigma^{ij} = C^{ijks}\varepsilon_{ks} \tag{3.4}$$

(C^{ijks} are the contravariant components of the fourth-rank tensor that defines the properties of the elastic medium).

It is convenient to write relations (3.4) in vector form:

$$\hat{\mathbf{t}}^i = J\widetilde{C}^{ij} \cdot \mathbf{u}_{,j}. \tag{3.5}$$

Here $\widetilde{C}^{ij}$ is the operator given by the formula $\widetilde{C}^{ij} = C^{ijks}(\ni_k * \ni_s)$, where the asterisk denotes tensor multiplication.

For simplicity, we restrict further discussion to the case of boundary conditions where the boundary S of the deformed body is composed of two parts: S_u (displacements are specified)

$$\mathbf{u}|_{S_u} = \mathbf{u}_*; \tag{3.6}$$

and S_σ (stresses are specified)

$$\mathbf{t}^i\nu_i|_{S_\sigma} = \mathbf{P}_* \tag{3.7}$$

(ν_i are the cosines of the outward normal vector to the boundary S; and $\mathbf{u}_*$ and $\mathbf{P}_*$ are vector functions specified on S).

Equations (3.1) and (3.5) and boundary conditions (3.6) and (3.7) define the boundary-value problem of the linear theory of elasticity.

4. Expansion of Functions in Terms of Legendre's Polynomials. We select the coordinate system ξ^k of the layer as a curvilinear coordinate system. In this case, the coordinate $\xi^3 \in [-1, 1]$ and the unknown functions $\mathbf{u}$ and $\hat{\mathbf{t}}^i$ are represented as series in terms of Legendre's polynomials:

$$\mathbf{u} = \sum_{k=0}^{\infty}[\mathbf{u}]^k P_k, \qquad \hat{\mathbf{t}}^i = \sum_{k=0}^{\infty}[\hat{\mathbf{t}}^i]^k P_k. \tag{4.1}$$

Here $P_k(\xi^3)$ are the orthogonal Legendre's polynomials; and $[\mathbf{u}]^k$ and $[\hat{\mathbf{t}}^i]^k$ are the expansion coefficients that depend on the Gaussian coordinates $\{\xi^\alpha\} \in S_\xi \subset R^2$:

$$[\mathbf{u}]^k = \frac{1+2k}{2}\int_{-1}^{1} \mathbf{u}P_k d\xi^3, \qquad [\hat{\mathbf{t}}^i]^k = \frac{1+2k}{2}\int_{-1}^{1} \hat{\mathbf{t}}^i P_k d\xi^3.$$

Let us expand the quantities $\hat{\mathbf{t}}^i$ in terms of the main local basis $\mathbf{a}_i$. According to (2.4) and (3.1), we have

$$\hat{\mathbf{t}}^i = J\sigma^{ij}\ni_i = \sqrt{a}mh\,(\sigma^{i\alpha}\ni_\alpha + \sigma^{i3}\ni_3) = \sqrt{a}mh\,(\sigma^{i\alpha}m_\alpha^\beta \mathbf{a}_\beta + \sigma^{ik}g_{k3}\mathbf{a}_3),$$

from which, using the rule of index lowering, we obtain

$$\hat{\mathbf{t}}^i = \sqrt{a}mh\,(\sigma^{i\alpha}m_\alpha^\beta \mathbf{a}_\beta + \sigma_3^i \mathbf{n}/h). \tag{4.2}$$

Substituting (4.2) into formulas (4.1) for the expansion of $\hat{\mathbf{t}}^i$ in terms of Legendre's polynomials, after simple transformations, we have

$$\hat{\mathbf{t}}^i = \sqrt{a}h\sum_{k=0}^{\infty}\frac{1+2k}{2}\,(\overset{(k)}{M}{}^{i\gamma}\mathbf{a}_\gamma + \overset{(k)}{Q}{}^i\mathbf{n})\,P_k, \tag{4.3}$$

where

$$\overset{(k)}{M}{}^{i\gamma} = \int_{-1}^{1} mm_\beta^\gamma\sigma^{i\beta}P_k d\xi^3; \qquad \overset{(k)}{Q}{}^i = \frac{1}{h}\int_{-1}^{1} m\sigma_3^i P_k d\xi^3;$$

$\overset{(k)}{M}{}^{\alpha\beta}$ are the moments of the tangential stresses of the kth order; and $\overset{(k)}{Q}{}^{\alpha}$ are the moments of the transverse (shearing) forces of the kth order.

As with the forces, let us expand the displacement vector $\mathbf{u}$ in (4.1) in the main basis:

$$\mathbf{u} = \sum_{k=0}^{\infty} (\overset{(k)}{U}{}^{\gamma} \mathbf{a}_{\gamma} + \overset{(k)}{W} \mathbf{n}) P_k. \tag{4.4}$$

Here

$$\overset{(k)}{U}{}^{\gamma} = \frac{1+2k}{2} \int_{-1}^{1} (\mathbf{u} \cdot \mathbf{a}^{\gamma}) P_k d\xi^3, \qquad \overset{(k)}{W} = \frac{1+2k}{2} \int_{-1}^{1} (\mathbf{u} \cdot \mathbf{n}) P_k d\xi^3$$

are the moments of the tangential and transverse displacements of the kth order.

It has been shown [3] that the first terms $\overset{(0)}{M}{}^{\alpha\gamma}$, $\overset{(0)}{Q}{}^{\alpha}$, and $\overset{(1)}{M}{}^{\alpha\gamma}$ in the expansion of the stresses in terms of Legendre's polynomials (4.3) have the meaning of the forces and moments acting on an element of the layer, and that an arbitrary rigid displacement of the layer appears in the first terms $\overset{(0)}{U}{}^{\alpha}$, $\overset{(1)}{U}{}^{\alpha}$, and $\overset{(0)}{W}$ of the series of displacements (4.4). This imposes a natural restriction on the minimum number of terms in the truncated approximating series (4.1). Thus, if the moment state is taken into account, the number of terms in the truncated approximating series (4.1) for displacements cannot be less than two for tangential displacements and not less than unity for transverse ones.

Approximations of the quantities $\hat{\mathbf{t}}^i$ and $\mathbf{u}$ consist in truncating series (4.1) and in reducing the initial differential problem to the solution of a finite system of equations in two independent variables.

5. Approximation of Stresses. Let us consider the equations of equilibrium of a continuum in a form equivalent to (3.1):

$$\mathbf{n} \times (\hat{\mathbf{t}}^i_{,i} + \mathbf{f}) = 0, \qquad \mathbf{n} \cdot (\hat{\mathbf{t}}^i_{,i} + \hat{\mathbf{f}}) = 0, \tag{5.1}$$

where, as above, $\mathbf{n}$ is a unit vector along the ξ^3 axis. The vector $\mathbf{n}$ does not depend on the variable ξ^3. Therefore, expanding Eqs. (5.1) into Legendre's polynomial series, we obtain the system

$$\mathbf{n} \times ([\hat{\mathbf{t}}^{\alpha}]^k_{,\alpha} + [\hat{\mathbf{t}}^3_{,3}]^k + [\hat{\mathbf{f}}]^k) = 0 \quad (k = \overline{0, M}), \qquad \mathbf{n} \cdot ([\hat{\mathbf{t}}^{\alpha}]^k_{,\alpha} + [\hat{\mathbf{t}}^3_{,3}]^k + [\hat{\mathbf{f}}]^k) = 0 \quad (k = \overline{0, N}), \tag{5.2}$$

where M and N are arbitrary numbers. For each k we multiply Eqs. (5.2) by P_k and sum the results. As a result, we have

$$\mathbf{n} \times \widehat{\mathbf{T}}'^{\alpha}_{,\alpha} + \mathbf{n} \times \sum_{k=0}^{M} [\hat{\mathbf{t}}^3_{,3}]^k P_k + \mathbf{n} \times \widehat{\mathbf{F}} = 0, \qquad \mathbf{n} \cdot \widehat{\mathbf{T}}''^{\alpha}_{,\alpha} + \mathbf{n} \cdot \sum_{k=0}^{N} [\hat{\mathbf{t}}^3_{,3}]^k P_k + \mathbf{n} \cdot \widehat{\mathbf{F}} = 0.$$

Here the quantities $\widehat{\mathbf{T}}'^{\alpha}$, $\widehat{\mathbf{T}}''^{\alpha}$, and $\widehat{\mathbf{F}}$ stand for the truncated series:

$$\widehat{\mathbf{T}}'^{\alpha} = \sum_{k=0}^{M} [\hat{\mathbf{t}}^{\alpha}]^k P_k, \quad \widehat{\mathbf{T}}''^{\alpha} = \sum_{k=0}^{N} [\hat{\mathbf{t}}^{\alpha}]^k P_k, \quad \widehat{\mathbf{F}} = \mathbf{n} \times \Big(\sum_{k=0}^{M} [\hat{\mathbf{f}}]^k \times \mathbf{n} P_k \Big) + \mathbf{n} \cdot \Big(\sum_{k=0}^{N} [\hat{\mathbf{f}}]^k \cdot \mathbf{n} P_k \Big). \tag{5.3}$$

Let us consider the function $a(\xi)$, the truncated series $A(\xi) = \sum\limits_{k=0}^{Q} [a]^k P_k(\xi)$ corresponding to it, and the series for the derivative $A_{,\xi} = \sum\limits_{k=0}^{Q-1} [a_{,\xi}]^k P_k$. One can show that for $L \leqslant (Q-1)$ the following equality holds:

$$A^*_{,\xi} = \sum_{k=0}^{L} [a_{,\xi}]^k P_k. \tag{5.4}$$

Here

$$A^*(\xi) = \sum_{k=0}^{Q} [a]^k \overset{(L)}{P_k}(\xi);$$

and $\overset{(L)}{P_k}$ are Legendre's polynomials of the form

$$\overset{(L)}{P_k} = \begin{cases} P_k, & k = \overline{0,(L-1)}, \\ P_L, & k = L+2i, \\ P_{L+1}, & k = L+2i+1, \end{cases} \quad i = 0,\, 1,\, \ldots$$

In this case, we obtain $A^*(\pm 1) = A(\pm 1)$ at the ends of the truncated series $[-1,1]$. For an arbitrary function $b(\xi)$ one can show the validity of the equation

$$\int_{-1}^{1} (A^*_{,\xi} b)\, d\xi = \int_{-1}^{1} (A^* b)_{,\xi} d\xi - \int_{-1}^{1} (aB^*_{,\xi})\, d\xi. \tag{5.5}$$

Here

$$B^*(\xi) = \sum_{k=0}^{Q+1} [b]^k \overset{(QL)}{P_k}(\xi); \tag{5.6}$$

the polynomials $\overset{(QL)}{P_k}$ for even values of $(Q-L)$ are

$$\overset{(QL)}{P_k} = \begin{cases} P_k, & k = \overline{0,L}, \\ P_{Q+1}, & k = L+2i+1, \quad i = \overline{0,(Q-L)/2}, \\ P_Q, & k = L+2i+2, \quad i = \overline{0,(Q-L)/2-1}, \end{cases}$$

and for odd values are

$$\overset{(QL)}{P_k} = \begin{cases} P_k, & k = \overline{0,L}, \\ P_Q, & k = L+2i+1, \\ P_{Q+1}, & k = L+2i+2, \end{cases} \quad i = \overline{0,(Q-L-1)/2}.$$

For the truncated series $B(\xi) = \sum_{k=0}^{Q+1} [b]^k P_k(\xi)$ and B^*, it follows from (5.6) that $B^*(\pm 1) = B(\pm 1)$. Using (5.4) we find that for the truncated series

$$\hat{\mathbf{T}}^3 = \mathbf{n} \times \Big(\sum_{k=0}^{M^*} \{[\hat{\mathbf{t}}^3]^k P_k\} \times \mathbf{n} \Big) + \mathbf{n} \cdot \Big(\sum_{k=0}^{N^*} \{[\hat{\mathbf{t}}^3]^k P_k\} \cdot \mathbf{n} \Big)$$

with arbitrary M^* and N^* satisfying the conditions $M^* \geqslant M+1$ and $N^* \geqslant N+1$, the equalities

$$\mathbf{n} \times \sum_{k=0}^{M} [\hat{\mathbf{t}}^3_{,3}]^k P_k = \mathbf{n} \times \hat{\mathbf{T}}^*_{,3}, \qquad \mathbf{n} \cdot \sum_{k=0}^{N} [\hat{\mathbf{t}}^3_{,3}]^k P_k = \mathbf{n} \cdot \hat{\mathbf{T}}^*_{,3}, \tag{5.7}$$

are valid, where

$$\hat{\mathbf{T}}^* = \mathbf{n} \times \Big(\sum_{k=0}^{M^*} \{[\hat{\mathbf{t}}^3]^k \overset{(M)}{P_k}\} \times \mathbf{n} \Big) + \mathbf{n} \cdot \Big(\sum_{k=0}^{N^*} \{[\hat{\mathbf{t}}^3]^k \overset{(N)}{P_k}\} \cdot \mathbf{n} \Big). \tag{5.8}$$

Substituting (5.7) into (5.2), we find

$$\mathbf{n} \times \hat{\mathbf{T}}'^i_{,i} + \mathbf{n} \times \hat{\mathbf{F}} = 0, \qquad \mathbf{n} \cdot \hat{\mathbf{T}}''^i_{,i} + \mathbf{n} \cdot \hat{\mathbf{F}} = 0. \tag{5.9}$$

Here, for brevity, we introduce the notation $\hat{\mathbf{T}}'^3 = \hat{\mathbf{T}}''^3 = \hat{\mathbf{T}}^*$.

Thus, in Eqs. (5.9) we have two types of approximations $\hat{\mathbf{T}}'^\alpha$ and $\hat{\mathbf{T}}''^\alpha$ for the same values of $\hat{\mathbf{t}}^\alpha$, which differ only in the number of terms retained in the series.

Substituting expressions (4.3) into (5.3) and (5.8), we obtain the final form of approximations for the stresses:

$$\widehat{\mathbf{T}}'^{\alpha} = \sqrt{a}h \sum_{k=0}^{M} \frac{1+2k}{2} \left(\overset{(k)}{M}{}^{\alpha\gamma} \mathbf{a}_{\gamma} + \overset{(k)}{Q}{}^{\alpha} \mathbf{n} \right) P_k, \quad \widehat{\mathbf{T}}''^{\alpha} = \sqrt{a}h \sum_{k=0}^{N} \frac{1+2k}{2} \left(\overset{(k)}{M}{}^{\alpha\gamma} \mathbf{a}_{\gamma} + \overset{(k)}{Q}{}^{\alpha} \mathbf{n} \right) P_k,$$
$$\widehat{\mathbf{T}}^{3} = \sqrt{a}h \Big(\sum_{k=0}^{M^*} \frac{1+2k}{2} \overset{(k)}{M}{}^{3\gamma} \mathbf{a}_{\gamma} P_k + \sum_{k=0}^{N^*} \frac{1+2k}{2} \overset{(k)}{Q}{}^{3} \mathbf{n} P_k \Big), \quad M^* \geqslant M+1, \quad N^* \geqslant N+1. \tag{5.10}$$

6. Approximations of Deformations and Displacements. Let us consider an arbitrary displacement vector $\mathbf{u}$ that satisfies conditions (3.6) at the boundary S_u. For simplicity, we consider only the case of zero volume forces ($\widehat{\mathbf{F}} = 0$).

It follows from Eqs. (5.9) that

$$\int_{V_\xi} \{(\widehat{\mathbf{T}}'^{i}_{,i} \times \mathbf{n}) \cdot (\mathbf{u} \times \mathbf{n}) + (\widehat{\mathbf{T}}''^{i}_{,i} \cdot \mathbf{n})(\mathbf{u} \cdot \mathbf{n})\}\, dV_\xi = 0, \qquad dV_\xi = d\xi^1 d\xi^2 d\xi^3. \tag{6.1}$$

Integrating (6.1) by parts, we get

$$\int_{V_\xi} \{[\widehat{\mathbf{T}}'^{i} \cdot (\mathbf{n} \times (\mathbf{u} \times \mathbf{n}))]_{,i} + [(\widehat{\mathbf{T}}''^{i} \cdot \mathbf{n})(\mathbf{u} \cdot \mathbf{n})]_{,i}\}\, dV_\xi = \int_{V_\xi} \{\widehat{\mathbf{T}}'^{i} \cdot [\mathbf{n} \times (\mathbf{u} \times \mathbf{n})]_{,i} + \widehat{\mathbf{T}}''^{i} \cdot [\mathbf{n} \cdot (\mathbf{u} \cdot \mathbf{n})]_{,i}\}\, dV_\xi. \tag{6.2}$$

Transforming the right-hand side (RS) of (6.2), we obtain

$$\mathrm{RS} = \int_{V_\xi} \{\widehat{\mathbf{T}}'^{i} \cdot [\mathbf{n} \times (\mathbf{u} \times \mathbf{n})]_{,i} + \widehat{\mathbf{T}}''^{i} \cdot [\mathbf{n} \cdot (\mathbf{u} \cdot \mathbf{n})]_{,i}\}\, dV_\xi$$
$$= \int_{V_\xi} \{\widehat{\mathbf{T}}'^{\alpha} \cdot [\mathbf{n} \times (\mathbf{u} \times \mathbf{n})]_{,\alpha} + \widehat{\mathbf{T}}''^{\alpha} \cdot [\mathbf{n} \cdot (\mathbf{u} \cdot \mathbf{n})]_{,\alpha} + \widehat{\mathbf{T}}^{3} \cdot \mathbf{u}_{,3}\}\, dV_\xi$$
$$= \int_{V_\xi} \Big\{ \hat{\mathbf{t}}^{\alpha} \cdot \Big[\mathbf{n} \times \Big(\sum_{k=0}^{M} [\mathbf{u}]^k P_k \times \mathbf{n} \Big)\Big]_{,\alpha} + \hat{\mathbf{t}}^{\alpha} \cdot \Big[\mathbf{n} \cdot \Big(\sum_{k=0}^{N} [\mathbf{u}]^k P_k \cdot \mathbf{n} \Big)\Big]_{,\alpha} + \hat{\mathbf{t}}^{3} \cdot \mathbf{U}''_{,3} \Big\}\, dV_\xi = \int_{V_\xi} \{\hat{\mathbf{t}}^{\alpha} \cdot \mathbf{U}'_{,\alpha} + \hat{\mathbf{t}}^{3} \cdot \mathbf{U}''_{,3}\}\, dV_\xi. \tag{6.3}$$

Here

$$\mathbf{U}' = \sum_{k=0}^{M} (\mathbf{n} \times ([\mathbf{u}]^k \times \mathbf{n}))\, P_k + \sum_{k=0}^{N} (\mathbf{n} \cdot ([\mathbf{u}]^k \cdot \mathbf{n}))\, P_k;$$
$$\mathbf{U}'' = \sum_{k=0}^{M^*+1} (\mathbf{n} \times ([\mathbf{u}]^k \times \mathbf{n}))\, \overset{(M^*M)}{P_k} + \sum_{k=0}^{N^*+1} (\mathbf{n} \cdot ([\mathbf{u}]^k \cdot \mathbf{n}))\, \overset{(N^*N)}{P_k}.$$

The expressions for $\mathbf{U}''$ are derived using relations (5.5) and (5.6). Substituting relations (3.1) for $\hat{\mathbf{t}}^i$ into (6.3) and using the symmetry of the stress tensor, we perform the transformation

$$\mathrm{RS} = \int_{V_\xi} \{\hat{\mathbf{t}}^{\alpha} \cdot \mathbf{U}'_{,\alpha} + \hat{\mathbf{t}}^{3} \cdot \mathbf{U}''_{,3}\}\, dV_\xi = \int_{V} \{\sigma^{\alpha k} \mathbf{э}_k \cdot \mathbf{U}'_{,\alpha} + \sigma^{3k} \mathbf{э}_k \cdot \mathbf{U}''_{,3}\}\, dV$$
$$= \int_{V} \{\sigma^{\alpha\beta} 0{,}5\, [\mathbf{э}_\beta \cdot \mathbf{U}'_{,\alpha} + \mathbf{э}_\alpha \cdot \mathbf{U}'_{,\beta}] + \sigma^{3\alpha} [\mathbf{э}_\alpha \cdot \mathbf{U}''_{,3} + \mathbf{э}_3 \cdot \mathbf{U}'_{,\alpha}] + \sigma^{33} [\mathbf{э}_3 \cdot \mathbf{U}''_{,3}]\}\, dV \quad (dV = J\, d\xi^1 d\xi^2 d\xi^3).$$

Denoting the expressions in square brackets by E_{ij}, we obtain

$$2E_{\alpha\beta} = \mathbf{э}_\beta \cdot \mathbf{U}'_{,\alpha} + \mathbf{э}_\alpha \cdot \mathbf{U}'_{,\beta}, \quad 2E_{3\alpha} = \mathbf{э}_\alpha \cdot \mathbf{U}''_{,3} + \mathbf{э}_3 \cdot \mathbf{U}'_{,\alpha}, \quad E_{33} = \mathbf{э}_3 \cdot \mathbf{U}''_{,3}; \tag{6.4}$$

and, finally, we have

$$\mathrm{RS} = \int_{V} \sigma^{ij} E_{ij}\, dV. \tag{6.5}$$

Comparing (6.4) with the expressions for strains in terms of displacements (3.3), we assume that the quantities E_{ij} are approximations of the strains ε_{ij} as truncated series in terms of Legendre's polynomials and that the vectors $\mathbf{U}'$ and $\mathbf{U}''$ are two approximations of the displacement vector $\mathbf{u}$; one corresponds to the derivatives with respect to the coordinates ξ^α, and the other, to the derivative with respect to the coordinate ξ^3.

7. Approximation of the Boundary Conditions. Integrating the left-hand side (LS) of (6.2), we obtain

$$\mathrm{LS} = \int_\Sigma \{\widehat{\mathbf{T}}'^1 \cdot (\mathbf{n} \times (\mathbf{u} \times \mathbf{n})) + (\widehat{\mathbf{T}}''^1 \cdot \mathbf{n})(\mathbf{u} \cdot \mathbf{n})\}\, d\xi^2 d\xi^3 + \int_\Sigma \{\widehat{\mathbf{T}}'^2 \cdot (\mathbf{n} \times (\mathbf{u} \times \mathbf{n}))$$
$$+ (\widehat{\mathbf{T}}''^2 \cdot \mathbf{n})(\mathbf{u} \cdot \mathbf{n})\}\, d\xi^1 d\xi^3 + \int_{S^+} \widehat{\mathbf{T}}^3 \cdot \mathbf{u}\, d\xi^1 d\xi^2 - \int_{S^-} \widehat{\mathbf{T}}^3 \cdot \mathbf{u}\, d\xi^1 d\xi^2. \tag{7.1}$$

We estimate the sum of the first two integrals in (7.1). For this purpose, using the orthogonality of Legendre's polynomials, we substitute the corresponding truncated series $\mathbf{U}'$ for the vector $\mathbf{u}$. Next, since Σ is a ruled surface, the following equalities hold:

$$d\xi^1 d\xi^3 = (\nu_2^0 / J^0)\, d\sigma^0, \qquad d\xi^2 d\xi^3 = (\nu_1^0 / J^0)\, d\sigma^0.$$

Here ν_α^0 are the cosines of the outward normal $\boldsymbol{\nu}^0$ to the side surface Σ at points of the boundary L; $J^0 = \mathbf{\ni}_1^0 \cdot (\mathbf{\ni}_2^0 \times \mathbf{\ni}_3^0)$; $d\sigma^0 = |d\mathbf{L} \times \mathbf{\ni}_3|\, d\xi^3$; and $d\mathbf{L}$ is an increment of the unit vector tangent to the curve L for motion counterclockwise along the boundary. As a result, for the sum of the first two integrals from (7.1) we obtain

$$\int_\Sigma \frac{\widehat{\mathbf{T}}^\alpha \cdot \mathbf{U}'}{J^0} \nu_\alpha^0 d\sigma^0 \qquad (\widehat{\mathbf{T}}^\alpha = \mathbf{n} \times (\widehat{\mathbf{T}}'^\alpha \times \mathbf{n}) + \mathbf{n} \cdot (\widehat{\mathbf{T}}''^\alpha \cdot \mathbf{n})).$$

In the last two integrals from (7.1), which are related to the faces S^+ and S^-, we replace the product $d\xi^1 d\xi^2$ according to the formula

$$d\xi^1 d\xi^2 = \left[\frac{\nu_3 dS}{J}\right]^+ = -\left[\frac{\nu_3 dS}{J}\right]^-,$$

where $\nu_3 = \boldsymbol{\nu} \cdot \mathbf{\ni}_3$; $\boldsymbol{\nu}$ is the outward normal to the surface S; the plus and minus signs correspond to the surfaces S^+ and S^-.

After the above transformations we reduce equality (7.1) to the form

$$\mathrm{LS} = \int_\Sigma \frac{\widehat{\mathbf{T}}^\alpha \cdot \mathbf{U}'}{J^0} \nu_\alpha^0 d\sigma^0 + \int_{S^+} \frac{\widehat{\mathbf{T}}^3 \cdot \mathbf{u}}{J} \nu_3 dS^+ + \int_{S^-} \frac{\widehat{\mathbf{T}}^3 \cdot \mathbf{u}}{J} \nu_3 dS^-. \tag{7.2}$$

An approximation of boundary conditions (3.6) and (3.7) by the following truncated series naturally follows from the first integral in (7.2):

$$\mathbf{U}'|_{\Sigma_u} = \mathbf{u}'_*; \tag{7.3}$$

$$\left.\frac{\widehat{\mathbf{T}}^\alpha \nu_\alpha^0}{J^0}\right|_{\Sigma_\sigma} = \mathbf{P}'_* \qquad (\Sigma_u \cup \Sigma_\sigma = \Sigma). \tag{7.4}$$

Here

$$\mathbf{u}'_* = \sum_{k=0}^{M} (\mathbf{n} \times ([\mathbf{u}_*]^k \times \mathbf{n}))\, P_k + \sum_{k=0}^{N} (\mathbf{n} \cdot ([\mathbf{u}_*]^k \cdot \mathbf{n}))\, P_k;$$

$$\mathbf{P}'_* = \sum_{k=0}^{M} (\mathbf{n} \times ([\mathbf{P}_*]^k \times \mathbf{n}))\, P_k + \sum_{k=0}^{N} (\mathbf{n} \cdot ([\mathbf{P}_*]^k \cdot \mathbf{n}))\, P_k.$$

Next, we consider the faces S^+ and S^-. Following boundary conditions (3.6) and (3.7), the displacements $\mathbf{u}|_{S_u^+} = \mathbf{u}_*$ and $\mathbf{u}|_{S_u^-} = \mathbf{u}_*$ are specified on a part of the boundaries S_u^+ and S_u^-, while the stresses

$$\mathbf{t}^3\nu_3\Big|_{S_\sigma^+} = \mathbf{P}_*, \qquad \mathbf{t}^3\nu_3\Big|_{S_\sigma^-} = \mathbf{P}_* \tag{7.5}$$

are specified on S_σ^+ and S_σ^-. In the last two integrals on the right-hand side of (7.2), the quantity $\hat{\mathbf{T}}^3\nu_3/J$ represents surface forces. Therefore, it is natural to replace (7.5) by the following boundary conditions at S_σ^+ and S_σ^-:

$$\frac{\hat{\mathbf{T}}^3\nu_3}{J}\Big|_{S_\sigma^+} = \mathbf{P}_*, \qquad \frac{\hat{\mathbf{T}}^3\nu_3}{J}\Big|_{S_\sigma^-} = \mathbf{P}_*. \tag{7.6}$$

Since the vector $\mathbf{u}$ is arbitrarily selected, we specify the boundary conditions

$$\mathbf{U}''\Big|_{S_u^+} = \mathbf{u}_*, \qquad \mathbf{U}''\Big|_{S_u^-} = \mathbf{u}_* \tag{7.7}$$

on the faces S_σ^+ and S_σ^-. Finally, using (7.3), (7.4), (7.6), and (7.7), Eq. (7.2) is written as

$$\mathrm{LS} = \int\limits_{\Sigma_\sigma}\mathbf{P}'_*\cdot\mathbf{U}'d\sigma^0 + \int\limits_{\Sigma_u}\frac{\hat{\mathbf{T}}^\alpha\cdot\mathbf{U}'_*}{J^0}\nu_\alpha^0 d\sigma^0 + \int\limits_{S_\sigma^+}\mathbf{P}_*\cdot\mathbf{U}''dS^+ + \int\limits_{S_\sigma^-}\mathbf{P}_*\cdot\mathbf{U}''dS^- + \int\limits_{S_u^+}\frac{\hat{\mathbf{T}}^3\cdot\mathbf{u}_*}{J}\nu_3 dS^+ + \int\limits_{S_u^-}\frac{\hat{\mathbf{T}}^3\cdot\mathbf{u}_*}{J}\nu_3 dS^-. \tag{7.8}$$

Correlating the left- and right-hand sides of Eq. (6.2), which are derived from (6.5) and (7.8), we obtain

$$\int\limits_V \sigma^{ij}E_{ij}dV = \int\limits_{\Sigma_\sigma}\mathbf{P}'_*\cdot\mathbf{U}'d\sigma^0 + \int\limits_{\Sigma_u}\frac{\hat{\mathbf{T}}^\alpha\cdot\mathbf{U}'_*}{J^0}\nu_\alpha^0 d\sigma^0 + \int\limits_{S_\sigma^+}\mathbf{P}_*\cdot\mathbf{U}''dS^+$$
$$+ \int\limits_{S_\sigma^-}\mathbf{P}_*\cdot\mathbf{U}''dS^- + \int\limits_{S_u^+}\frac{\hat{\mathbf{T}}^3\cdot\mathbf{u}_*}{J}\nu_3 dS^+ + \int\limits_{S_u^-}\frac{\hat{\mathbf{T}}^3\cdot\mathbf{u}_*}{J}\nu_3 dS^-. \tag{7.9}$$

Relation (7.9) is the condition of equality (balance) of the work of the external and internal forces.

8. Hooke's Law Approximation. We approximate Hooke's law (3.4) by the relations

$$\sigma^{ij} = C^{ijks}E_{ks}, \tag{8.1}$$

where E_{ks} are approximations of the strain tensor ε_{ks} (6.4):

$$2E_{\alpha\beta} = \mathbf{э}_\beta\cdot\mathbf{U}'_{,\alpha} + \mathbf{э}_\alpha\cdot\mathbf{U}'_{,\beta}, \qquad 2E_{3\alpha} = \mathbf{э}_\alpha\cdot\mathbf{U}''_{,3} + \mathbf{э}_3\cdot\mathbf{U}'_{,\alpha}, \qquad E_{33} = \mathbf{э}_3\cdot\mathbf{U}''_{,3}.$$

We present (8.1) in vector form similar to Eqs. (3.5) $\hat{\mathbf{t}}^i = J\sigma^{ij}\mathbf{э}_j = J(\tilde{C}^{i\alpha}\cdot\mathbf{U}'_{,\alpha} + \tilde{C}^{i3}\cdot\mathbf{U}''_{,3})$. Thus, for the coefficients of series (5.3) and (5.5), we obtain

$$[\hat{\mathbf{t}}^i]^k = \frac{1+2k}{2}\int\limits_{-1}^{1} J\,(\tilde{C}^{i\alpha}\cdot\mathbf{U}'_{,\alpha} + \tilde{C}^{i3}\cdot\mathbf{U}''_{,3})\,P_k d\xi^3. \tag{8.2}$$

9. System of Equations for (M,N)-Approximation. Using the above results, we write a system of two-dimensional equations. The lengths of the corresponding truncated series are specified by two pairs of numbers (M,N) and (M^*,N^*). As follows from (5.10), the inequalities

$$M^* \geqslant M+1, \qquad N^* \geqslant N+1 \tag{9.1}$$

should hold. It is natural, reasoning from constraint (9.1), to select minimum possible values of (M^*,N^*). In this case, however, specification of arbitrary conditions at the faces can affect the differential order of the equations. This is due to the symmetry condition for the stress tensor (3.2) $\mathbf{э}_i\times\hat{\mathbf{t}}^i = 0$, which can be written in the equivalent form $\mathbf{э}_\alpha\times\hat{\mathbf{t}}^\alpha = -h\mathbf{n}\times\hat{\mathbf{t}}^3$. Performing vector and scalar multiplication of the last equality by $\mathbf{n}$, after transformations we obtain

$$\mathbf{э}_\alpha\cdot(\mathbf{n}\times\hat{\mathbf{t}}^\alpha) = 0, \qquad \mathbf{n}\times(\mathbf{э}_\alpha\times\hat{\mathbf{t}}^\alpha) = h\mathbf{n}\times(\hat{\mathbf{t}}^3\times\mathbf{n}). \tag{9.2}$$

Let us consider the second equality of (9.2), since it is precisely this equality that contains the quantity $\hat{\mathbf{t}}^3$, which is responsible for assignment of stresses on the face surfaces [condition (7.6)]. For simplicity, we limit our study to plates of constant thickness, since in the more general case all arguments are similar. For plates of constant thickness we have

$$\mathsf{э}_\alpha = \mathsf{э}^0_\alpha, \qquad \mathbf{n}\cdot\mathsf{э}^0_\alpha = 0. \tag{9.3}$$

From (9.2) and (9.3), it follows that $\mathsf{э}^0_\alpha\cdot(\mathbf{n}\cdot\hat{\mathbf{t}}^\alpha) = h\mathbf{n}\times(\hat{\mathbf{t}}^3\times\mathbf{n})$, and for the coefficients of the series in terms of Legendre's polynomials, respectively,

$$\mathsf{э}^0_\alpha\cdot(\mathbf{n}\cdot[\hat{\mathbf{t}}^\alpha]^k) = h\mathbf{n}\times([\hat{\mathbf{t}}^3]^k\times\mathbf{n}). \tag{9.4}$$

The first derivatives of the quantities $(\mathbf{n}\cdot[\hat{\mathbf{t}}^\alpha]^k)$ enter into the equilibrium equations (5.2), and the parameter k takes values of from 0 to N. On the other hand, the products $\mathbf{n}\times([\hat{\mathbf{t}}^3]^k\times\mathbf{n})$ enter into the series for $\mathbf{T}^3$, and the parameter k takes values of from 0 to M^*. This series gives boundary conditions for stresses (7.6) at the faces. Consequently, it follows from (9.4) that for the differential order of the equations to be independent of assignment of the boundary conditions in stresses at the faces, the inequality

$$M^* \geqslant N+2. \tag{9.5}$$

should hold. Combining (9.1) and (9.5), we write a system of inequalities:

$$M^* \geqslant M+1, \qquad N^* \geqslant N+1, \qquad M^* \geqslant N+2. \tag{9.6}$$

Selecting the smallest values of the parameters M^* and N^* that satisfy inequalities (9.6), we have two possible variants:

$$\begin{aligned} M^* &= M+1, \qquad N^* = N+1, \qquad \text{if}\ \ M \geqslant N+1,\\ M^* &= N+2, \qquad N^* = N+1, \qquad \text{if}\ \ M \leqslant N+1. \end{aligned}$$

For $M = N+1$, we obtain a one-parameter family of N-approximations of the equations of an elastic layer of arbitrary thickness [3]. Thus, the lengths of all truncated series that enter into the equations are defined by assignment of two numbers M and N. The two-dimensional system of equations in (M,N)-approximation consists of:

the equilibrium equations [see (5.9)]

$$\mathbf{n}\times(\hat{\mathbf{T}}'^i_{,i}\times\mathbf{n}) + \mathbf{n}\cdot(\hat{\mathbf{T}}''^i_{,i}\cdot\mathbf{n}) = 0; \tag{9.7}$$

the equations of Hooke's law (8.2) in the form of series (5.3) and (5.8):

$$\begin{gathered}
\hat{\mathbf{T}}'^\alpha = \sum_{k=0}^{M} P_k \frac{1+2k}{2}\int_{-1}^{1} J\,(\tilde{C}^{\alpha\beta}\cdot\mathbf{U}'_{,\beta} + \tilde{C}^{\alpha 3}\cdot\mathbf{U}''_{,3})\,P_k d\xi^3,\\
\hat{\mathbf{T}}''^\alpha = \sum_{k=0}^{N} P_k \frac{1+2k}{2}\int_{-1}^{1} J\,(\tilde{C}^{\alpha\beta}\cdot\mathbf{U}'_{,\beta} + \tilde{C}^{\alpha 3}\cdot\mathbf{U}''_{,3})\,P_k d\xi^3,\\
\hat{\mathbf{T}}'^3 = \hat{\mathbf{T}}''^3 = \hat{\mathbf{T}}^* = \mathbf{n}\times\Big(\sum_{k=0}^{M^*} \overset{(M)}{P_k}\ \frac{1+2k}{2}\int_{-1}^{1} J\,(\tilde{C}^{3\beta}\cdot\mathbf{U}'_{,\beta} + \tilde{C}^{33}\cdot\mathbf{U}''_{,3})\times\mathbf{n}P_k d\xi^3\Big)\\
+\mathbf{n}\cdot\Big(\sum_{k=0}^{N^*} \overset{(N)}{P_k}\ \frac{1+2k}{2}\int_{-1}^{1} J\,(\tilde{C}^{3\beta}\cdot\mathbf{U}'_{,\beta} + \tilde{C}^{33}\cdot\mathbf{U}''_{,3})\cdot\mathbf{n}P_k d\xi^3\Big);
\end{gathered} \tag{9.8}$$

the conditions at the faces S^+ and S^- [see (7.6) and (7.7)]

$$\mathbf{U}''|_{S^+_u} = u_*, \qquad \mathbf{U}''|_{S^-_u} = u_*, \qquad \frac{\hat{\mathbf{T}}^*\nu_3}{J}\Big|_{S^+_\sigma} = \mathbf{P}_*, \qquad \frac{\hat{\mathbf{T}}^*\nu_3}{J}\Big|_{S^-_\sigma} = \mathbf{P}_*. \tag{9.9}$$

In determining the differential order of system (9.7)–(9.9) our reasoning is similar to [3] and is as follows. The strain–displacement relations (6.4) contain the coefficients of the series $\mathbf{U}'$ with their first-order partial derivatives with respect to the Gaussian coordinates ξ^α on the middle surface S^0, while the coefficients of the series $(\mathbf{U}'' - \mathbf{U}')$ occur without derivatives. The first and the second group of unknown coefficients are called basic and complementary, respectively. The complementary unknowns are found from Eqs. (9.9), which are the boundary conditions at the faces. These equations form a system of algebraic equations in the complementary unknowns. Solution of this system gives expressions for the complementary unknowns in terms of the basic unknowns.

Furthermore, if we insert these expressions into (9.8), we obtain formulas that relate the vector functions $\widehat{\mathbf{T}}'^\alpha$, $\widehat{\mathbf{T}}''^\alpha$, and $\widehat{\mathbf{T}}^*$ and the basic unknowns which are the coefficients of the series $\mathbf{U}'$. These formulas are linear forms with respect to the coefficients of the series $\mathbf{U}'$ and their first derivatives.

By inserting the expressions for $\widehat{\mathbf{T}}'^\alpha$, $\widehat{\mathbf{T}}''^\alpha$, and $\widehat{\mathbf{T}}^*$ into the equilibrium equations (9.7), we obtain a system of $2\,(M+1)+N+1$ scalar equations, each containing $2\,(M+1)+N+1$ scalar functions $(\mathbf{n}\times([u]^k\times\mathbf{n})$ $(k=\overline{0,M})$, $[u]^k\cdot\mathbf{n}\,(k=\overline{0,N}))$ together with their partial derivatives up to the second order inclusively. Thus, we have a $2n$th-order system to determine n functions, where

$$n = 2\,(M+1) + N + 1. \tag{9.10}$$

The differential order of the system for (M,N) approximation does not depend on the type of boundary conditions at the faces: either stresses or displacements can be specified.

For $M = 1$ and $N = 0$, we obtain the first approximation. In this case, it follows from (9.10) that $n = 5$, i.e., we have five basic unknowns: three displacements of the middle surface and two rotation angles. The corresponding differential order of system (9.7)–(9.10) is 10.

REFERENCES

1. I. N. Vekua, *Some General Methods for Constructing Different Variants of Shell Theory* [in Russian], Nauka, Moscow (1982).
2. G. V. Ivanov, *Theory of Plates and Shells* [in Russian], Izd. Novosib. Univ., Novosibirsk (1980).
3. A. E Alekseev, "Derivation of equations for a layer of variable thickness based on expansions in terms of Legendre's polynomials," *Prikl. Mekh. Tekh. Fiz.*, **35**, No. 4, 137–147 (1994).
4. B. L. Pelekh, A. V. Maksimuk, and I. M. Korovaichuk, *Contact Problems for Layered Members and Bodies with Coatings* [in Russian], Naukova Dumka, Kiev (1988).